2D Traffic Flow Modeling via Kinetic Models

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Scale-FreeBack

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Microscopic description (N-particle system)

$\dot{\mathbf{x}}_j = \mathbf{v}_j, \quad \dot{\mathbf{v}}_j = \mathbf{F}_j, \quad$ with force $\mathbf{F}_j = \mathbf{F}_j(\mathbf{x}_1, \dots, \mathbf{x}_N, \dot{\mathbf{x}}_1, \dots, \dot{\mathbf{x}}_N)$.

Kinetic (=mesoscopic) description

$\partial_t f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f + \mathbf{F} \cdot \nabla_{\mathbf{v}} f = C[f]$

Boltzmann equation (if collision term $C[f] = 0$: Vlasov equation)

$f = f(\mathbf{x}, t, \mathbf{v})$ phase space density

$=$ probability to find particle at position $\mathbf{x}$, time t, flying with velocity $\mathbf{v}$.

Macroscopic description

$\rho_t + \nabla \cdot (\rho \mathbf{u}) = 0$ conservation of mass (continuity equation)

$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = \dots$ balance of momentum

Key question

Can two particles at the same position have different velocities?

Yes: Kinetic description; No: Macroscopic description

Microscopic description

$\dot{\mathbf{x}}_j = \mathbf{v}_j, \quad \dot{\mathbf{v}}_j = \mathbf{F}_j.$

Kinetic description

$\partial_t f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f + \mathbf{F} \cdot \nabla_{\mathbf{v}} f = C[f]$

Macroscopic description

$\rho_t + \nabla \cdot (\rho \mathbf{u}) = 0, \quad \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = \dots$

Key question (slightly relaxed)

Can two particles at the same position have different velocities? Or:
Can two particles very close to each other have vastly different velocities?
Yes: Kinetic description required; No: Macroscopic does the job.

Example

$\mathbf{F}_j = 0$: free flight of particles without interaction ($\longrightarrow$ photons);
cannot be described macroscopically; need kinetic description.

1D Traffic Flow

Can two vehicles be at the same position, while having different velocities?

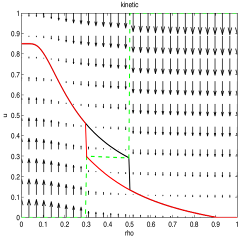
Strictly speaking: No.

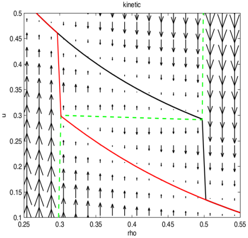
Yet, in lane-aggregated models: Yes (same x-position, but different lanes).

Traditional role of kinetic descriptions in traffic modeling

Not useful as stand-alone models.

But very useful to derive (new and existing) macroscopic models via a systematic procedure. Yields insight into the phenomenological connection between car-following dynamics and large-scale emergent phenomena.





[Klar, Guenther, Wegener, Materne, *SIAM J. Appl. Math.*, 2004]

[Borsche, Kimathi, Klar, *Computers Math. Appl.*, 2012]

[Visconti, *PhD thesis*, 2016]

2D Traffic Flow

Premise

- Describe large-scale flow of vehicles in large metropolitan area via a time-dependent PDE in 2D.

Advantages

- Conceptual simplification over flow on network.
- PDE natural for up- and down-scaling (lower computational cost).
- PDE framework allows for more direct application of control theory.

Key questions

- How many field quantities are (at minimum) needed to capture the true flow behavior of vehicles on the road network?
- Can one get away with a scalar (conservation) law for $\rho(x, y, t)$?
  Answer: Hard to conceive as this would not distinguish East-bound from West-bound traffic...

Four-Velocity Model (for Manhattan-type Cities)

Four 2D densities:

(i) $\rho_E(x,y)$: vehicles heading East (towards $x \to \infty$)

(ii) $\rho_W(x,y)$: vehicles heading West (towards $x \to -\infty$)

(iii) $\rho_N(x,y)$: vehicles heading North (towards $y \to \infty$)

(iv) $\rho_S(x,y)$: vehicles heading South (towards $y \to -\infty$).

Defined as

$$\rho_E(x,y) \approx \frac{\#\text{vehicles in } C_h(x,y) \text{ heading East}}{h^2}$$

where $C_h(x,y) = [x - \frac{h}{2}, x + \frac{h}{2}] \times [y - \frac{h}{2}, y + \frac{h}{2}]$
and h reasonable length scale.

Simplest Model

Vehicles move with a fixed speed $s > 0$ in their direction of heading. Moreover, vehicles never change their heading, and do not impede each other.

On domain $\Omega = [x_\mathrm{L}, x_\mathrm{R}] \times [y_\mathrm{L}, y_\mathrm{R}]$, this situation is described via the system of PDE

$$\begin{aligned}
\partial_t \rho_\mathrm{E} + s\,\partial_x \rho_\mathrm{E} &= 0 \qquad \text{with b.c. } \rho_\mathrm{E} = \rho_\mathrm{E}^\mathrm{in} \text{ for } x = x_\mathrm{L} \\
\partial_t \rho_\mathrm{W} - s\,\partial_x \rho_\mathrm{W} &= 0 \qquad \text{with b.c. } \rho_\mathrm{W} = \rho_\mathrm{W}^\mathrm{in} \text{ for } x = x_\mathrm{R} \\
\partial_t \rho_\mathrm{N} + s\,\partial_y \rho_\mathrm{N} &= 0 \qquad \text{with b.c. } \rho_\mathrm{N} = \rho_\mathrm{N}^\mathrm{in} \text{ for } y = y_\mathrm{L} \\
\partial_t \rho_\mathrm{S} - s\,\partial_y \rho_\mathrm{S} &= 0 \qquad \text{with b.c. } \rho_\mathrm{S} = \rho_\mathrm{S}^\mathrm{in} \text{ for } y = y_\mathrm{R}
\end{aligned}$$

For low densities (free-flow traffic), this linear transport model is actually not too bad (see later).

Changes of Direction

Real vehicles may take turns at intersections.

2D continuum description: changes in heading occur at any point (x, y).

$$\partial_t \rho_{\mathsf{E}} + s\,\partial_x \rho_{\mathsf{E}} = -\alpha_{\mathsf{E}\to\mathsf{N}}\rho_{\mathsf{E}} - \alpha_{\mathsf{E}\to\mathsf{S}}\rho_{\mathsf{E}} + \alpha_{\mathsf{N}\to\mathsf{E}}\rho_{\mathsf{N}} + \alpha_{\mathsf{S}\to\mathsf{E}}\rho_{\mathsf{S}}$$
$$\partial_t \rho_{\mathsf{W}} - s\,\partial_x \rho_{\mathsf{W}} = -\alpha_{\mathsf{W}\to\mathsf{N}}\rho_{\mathsf{W}} - \alpha_{\mathsf{W}\to\mathsf{S}}\rho_{\mathsf{W}} + \alpha_{\mathsf{N}\to\mathsf{W}}\rho_{\mathsf{N}} + \alpha_{\mathsf{S}\to\mathsf{W}}\rho_{\mathsf{S}}$$
$$\partial_t \rho_{\mathsf{N}} + s\,\partial_y \rho_{\mathsf{N}} = -\alpha_{\mathsf{N}\to\mathsf{E}}\rho_{\mathsf{N}} - \alpha_{\mathsf{N}\to\mathsf{W}}\rho_{\mathsf{N}} + \alpha_{\mathsf{E}\to\mathsf{N}}\rho_{\mathsf{E}} + \alpha_{\mathsf{W}\to\mathsf{N}}\rho_{\mathsf{W}}$$
$$\partial_t \rho_{\mathsf{S}} - s\,\partial_y \rho_{\mathsf{S}} = -\alpha_{\mathsf{S}\to\mathsf{E}}\rho_{\mathsf{S}} - \alpha_{\mathsf{S}\to\mathsf{W}}\rho_{\mathsf{S}} + \alpha_{\mathsf{E}\to\mathsf{S}}\rho_{\mathsf{E}} + \alpha_{\mathsf{W}\to\mathsf{S}}\rho_{\mathsf{W}}$$

Turning rates $\alpha_{\mathsf{old}\to\mathsf{new}}$: rate of vehicles heading in direction "old" switching to direction "new". Model parameters, same idea as split or turning ratios at off-ramps in 1D network models.

Generally $\alpha(x, t)$, or even $\alpha(x, t, \rho_j)$ (fewer left-turns possible if opposing flow at high density).

Kinetic Description

Can introduce additional directions (diagonal roads), all the way to a continuum of directions.

Yields kinetic model: direction of travel = polar angle θ w.r.t. East.

Phase space density of vehicles $\phi(t, x, y, \theta)$.

$$\partial_t \phi + s\cos(\theta)\partial_x \phi + s\sin(\theta)\partial_y \phi = \int_0^{2\pi} \sigma(t, x, y, \theta', \theta)\phi(t, x, y, \theta')\,\mathrm{d}\theta' - \int_0^{2\pi} \sigma(t, x, y, \theta, \theta')\,\mathrm{d}\theta'\;\phi(t, x, y, \theta)$$

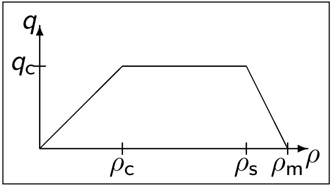
Here $\sigma(t, x, y, \theta', \theta)$ is **"collision" kernel** = probability that a vehicle heading in the direction θ' changes its heading to the new direction θ.

Multi-density PDE model is discretization (in θ) of kinetic model via finite set of directions.

Density Propagation in 1D

Long road with many traffic lights. Want large-scale averaged law of traffic flow: $\rho_t + q_x = 0$.

- Low density: Negligible queuing at lights. Vehicles advance at effective average speed s_f (depending on signal phase properties) independent of ρ. Hence $q = s_f\rho$. Macroscopic "free" flow.



- Medium density: Traffic lights limit flux. Green light lets 1 vehicle pass per time τ. Average (long time scale) throughput independent of ρ. Hence $q = q_c \approx \beta/\tau$, where $0 \leq \beta \leq 1$ is green portion.
- High density: Spillback effects (from an intersection to the one upstream) limit throughput.

In practice, rather than modeling from first principles, these large-scale fundamental diagrams should be determined from data.

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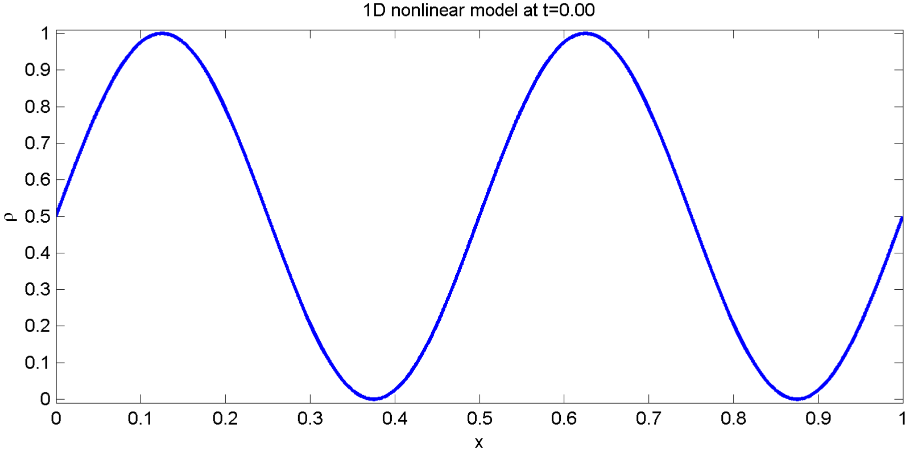
Density Propagation in 1D: Solution Behavior



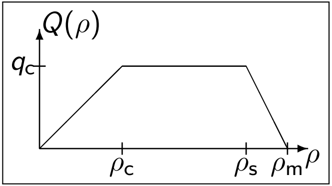
$\rho_c = 0.4,\ \rho_s = 0.8,\ \rho_m = 1,\ s_f = 1$

2D Model with Nonlinear 1D Fluxes

Yields nonlinear model with diagonal fluxes:

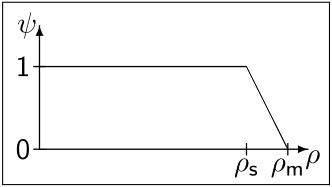
$$\begin{aligned}
\partial_t \rho_{\mathsf{E}} + \partial_x Q(\rho_{\mathsf{E}}) &= -\alpha_{\mathsf{E}\to\mathsf{N}}\rho_{\mathsf{E}} - \alpha_{\mathsf{E}\to\mathsf{S}}\rho_{\mathsf{E}} + \alpha_{\mathsf{N}\to\mathsf{E}}\rho_{\mathsf{N}} + \alpha_{\mathsf{S}\to\mathsf{E}}\rho_{\mathsf{S}} \\
\partial_t \rho_{\mathsf{W}} - \partial_x Q(\rho_{\mathsf{W}}) &= -\alpha_{\mathsf{W}\to\mathsf{N}}\rho_{\mathsf{W}} - \alpha_{\mathsf{W}\to\mathsf{S}}\rho_{\mathsf{W}} + \alpha_{\mathsf{N}\to\mathsf{W}}\rho_{\mathsf{N}} + \alpha_{\mathsf{S}\to\mathsf{W}}\rho_{\mathsf{S}} \\
\partial_t \rho_{\mathsf{N}} + \partial_y Q(\rho_{\mathsf{N}}) &= -\alpha_{\mathsf{N}\to\mathsf{E}}\rho_{\mathsf{N}} - \alpha_{\mathsf{N}\to\mathsf{W}}\rho_{\mathsf{N}} + \alpha_{\mathsf{E}\to\mathsf{N}}\rho_{\mathsf{E}} + \alpha_{\mathsf{W}\to\mathsf{N}}\rho_{\mathsf{W}} \\
\partial_t \rho_{\mathsf{S}} - \partial_y Q(\rho_{\mathsf{S}}) &= -\alpha_{\mathsf{S}\to\mathsf{E}}\rho_{\mathsf{S}} - \alpha_{\mathsf{S}\to\mathsf{W}}\rho_{\mathsf{S}} + \alpha_{\mathsf{E}\to\mathsf{S}}\rho_{\mathsf{E}} + \alpha_{\mathsf{W}\to\mathsf{S}}\rho_{\mathsf{W}}
\end{aligned}$$

In general, flux function Q depends on position (x, y), time t, and on the direction of travel: Q_{E}, Q_{W}, Q_{N}, and Q_{S}.



Modeling Spillback

Highly congested traffic: throughput reduction on city network due to spillback; one flow direction impedes another flow direction due to vehicles blocking the intersection. Coupling between fluxes at high densities.



Exceeding spillback density ($\rho > \rho_s$), flow impedes other directions. "Friction" function

$$\psi(\rho) = \min\left(\max\left(1 - \frac{\rho-\rho_s}{\rho_m-\rho_s}, 0\right), 1\right)$$

Example

Only ρ_E and ρ_N present:

$$Q_E(\rho_E, \rho_N) = Q(\rho_E)\psi(\rho_N)$$
$$Q_N(\rho_E, \rho_N) = Q(\rho_N)\psi(\rho_E)$$

with $Q(\rho)$ is 1D flux function.

All four directions present

$$Q_E = Q(\rho_E)\min\{\psi(\rho_N), \psi(\rho_S)\}$$
$$Q_W = Q(\rho_W)\min\{\psi(\rho_N), \psi(\rho_S)\}$$
$$Q_N = Q(\rho_N)\min\{\psi(\rho_E), \psi(\rho_W)\}$$
$$Q_S = Q(\rho_S)\min\{\psi(\rho_E), \psi(\rho_W)\}$$

Only perpendicular fluxes are reduced.

Modeling Impedance of Turning Rates

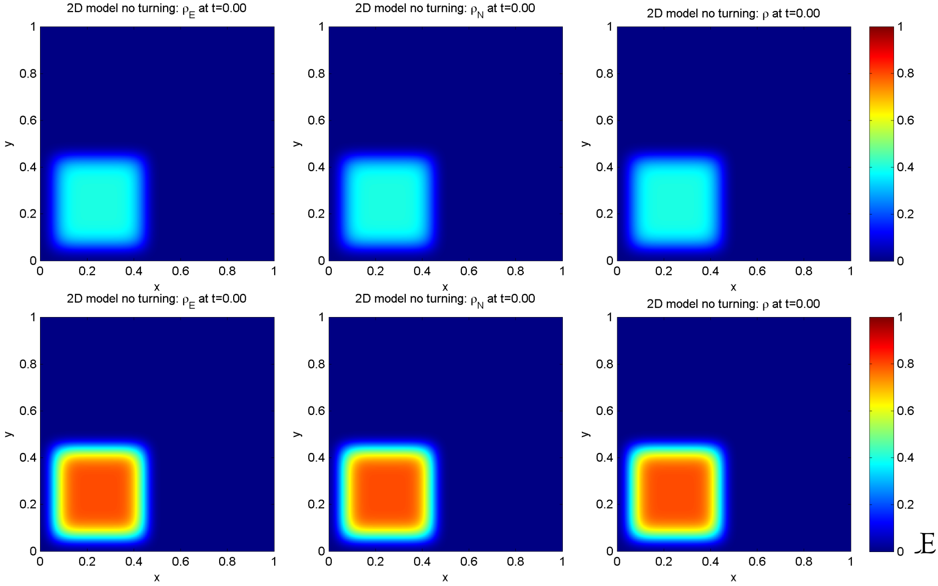
- Right turn: Friction from new direction of travel. Replace $\alpha_{\mathsf{E}\to\mathsf{S}}$ by
$$\tilde{\alpha}_{\mathsf{E}\to\mathsf{S}} = \alpha_{\mathsf{E}\to\mathsf{S}}\psi(\rho_{\mathsf{S}})\ .$$
- Left turn: $\alpha_{\mathsf{E}\to\mathsf{N}}$ receives friction from ρ_{N} (new direction), from ρ_{S} (squeezing through cars blocking the intersection), and from ρ_{W} (if no specific left turn phase):
$$\tilde{\alpha}_{\mathsf{E}\to\mathsf{N}} = \alpha_{\mathsf{E}\to\mathsf{N}} \min\{\psi(\rho_{\mathsf{N}}),\psi(\rho_{\mathsf{S}}),\psi(\rho_{\mathsf{W}})\}\ .$$
[Could also use different friction functions for different effects.]

Full model

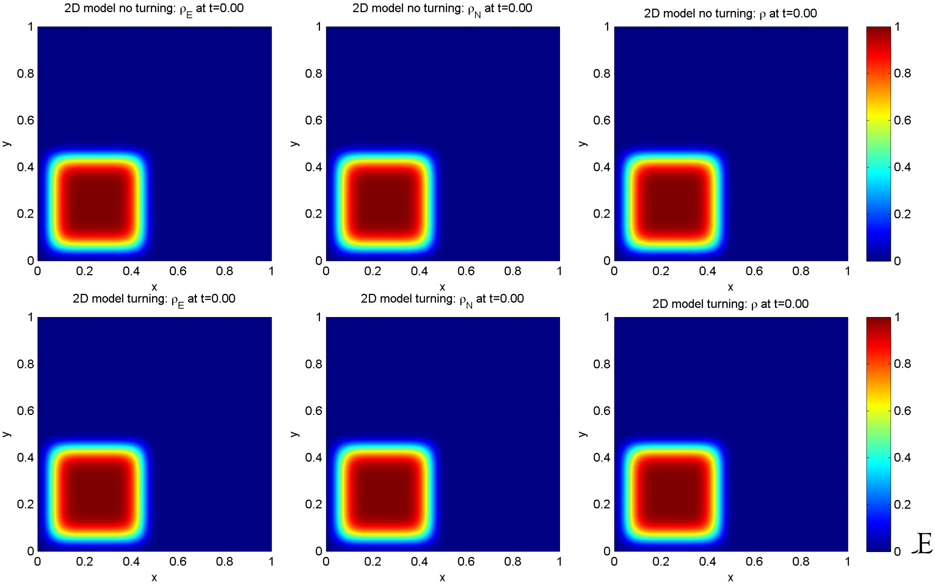
$$\begin{aligned}
\partial_t\rho_{\mathsf{E}} + \partial_x Q_{\mathsf{E}}(\rho) &= -\tilde{\alpha}_{\mathsf{E}\to\mathsf{N}}(\rho)\rho_{\mathsf{E}} - \tilde{\alpha}_{\mathsf{E}\to\mathsf{S}}(\rho)\rho_{\mathsf{E}} + \tilde{\alpha}_{\mathsf{N}\to\mathsf{E}}(\rho)\rho_{\mathsf{N}} + \tilde{\alpha}_{\mathsf{S}\to\mathsf{E}}(\rho)\rho_{\mathsf{S}}\\
\partial_t\rho_{\mathsf{W}} - \partial_x Q_{\mathsf{W}}(\rho) &= -\tilde{\alpha}_{\mathsf{W}\to\mathsf{N}}(\rho)\rho_{\mathsf{W}} - \tilde{\alpha}_{\mathsf{W}\to\mathsf{S}}(\rho)\rho_{\mathsf{W}} + \tilde{\alpha}_{\mathsf{N}\to\mathsf{W}}(\rho)\rho_{\mathsf{N}} + \tilde{\alpha}_{\mathsf{S}\to\mathsf{W}}(\rho)\rho_{\mathsf{S}}\\
\partial_t\rho_{\mathsf{N}} + \partial_x Q_{\mathsf{N}}(\rho) &= -\tilde{\alpha}_{\mathsf{N}\to\mathsf{E}}(\rho)\rho_{\mathsf{N}} - \tilde{\alpha}_{\mathsf{N}\to\mathsf{W}}(\rho)\rho_{\mathsf{N}} + \tilde{\alpha}_{\mathsf{E}\to\mathsf{N}}(\rho)\rho_{\mathsf{E}} + \tilde{\alpha}_{\mathsf{W}\to\mathsf{N}}(\rho)\rho_{\mathsf{W}}\\
\partial_t\rho_{\mathsf{S}} - \partial_x Q_{\mathsf{S}}(\rho) &= -\tilde{\alpha}_{\mathsf{S}\to\mathsf{E}}(\rho)\rho_{\mathsf{S}} - \tilde{\alpha}_{\mathsf{S}\to\mathsf{W}}(\rho)\rho_{\mathsf{S}} + \tilde{\alpha}_{\mathsf{E}\to\mathsf{S}}(\rho)\rho_{\mathsf{E}} + \tilde{\alpha}_{\mathsf{W}\to\mathsf{S}}(\rho)\rho_{\mathsf{W}}\ .
\end{aligned}$$

Here $\rho = (\rho_{\mathsf{E}},\rho_{\mathsf{W}},\rho_{\mathsf{N}},\rho_{\mathsf{S}})^T$.

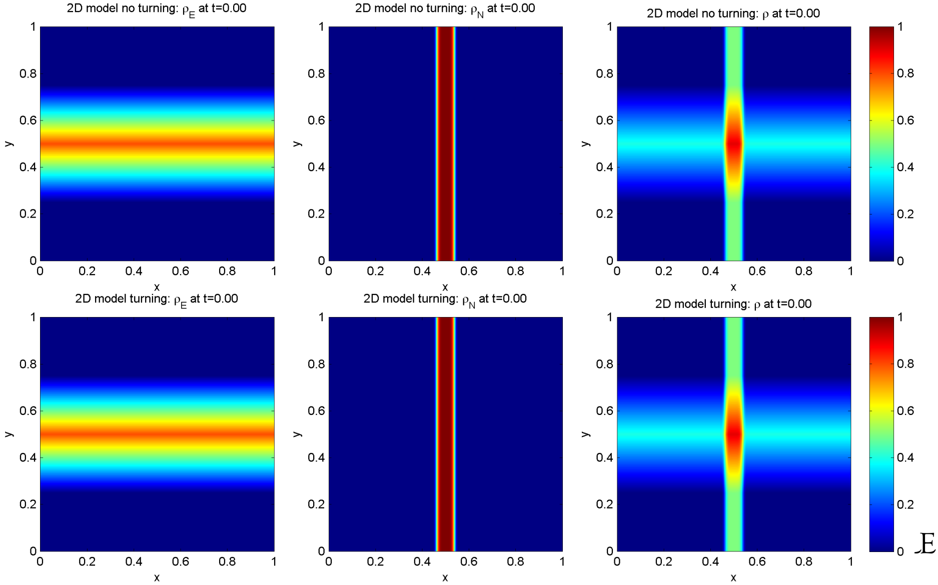
Model Results: No Interaction Between Directions



Model Results: Interaction Between Directions



Model Results: Blocking of Roads



Model Calibration for Kinetic 2d Traffic Models

- Conclusion: These types of models look intriguing, worth investigating further.
- Lots of questions, most fundamentally: Coupled models well-posed?
- A key open research step is how to calibrate (data-fit) and also validate these models.
- At first glance, this looks extremely challenging, given the high-dimensional structure of the scattering kernel $\sigma(t, x, y, \theta', \theta)$.
- However, (1) data are abundant to fit 5D functions; and
- (2) The 2D PDE model structure is actually really nice: We have removed the nasty network from the equations...
- ...Hence, rather than equipping roads and intersections with sensors, we can directly use vehicle trajectory data (GPS) in (longitude, latitude); no mapping onto the road network needed.

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