Reduction of network models of physical systems

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Outline

1. Introduction
2. Physical network dynamics: first order-case
3. Model reduction of mass-damper(-like) systems by clustering
4. Clustering based on balancing of edge activities
5. Kron-based model reduction of mass-damper(-like) systems
6. The non-symmetric case
7. Extension to mass-spring-damper systems
8. Conclusions and Outlook

Motivation

Large-scale physical network dynamics appear in many areas:

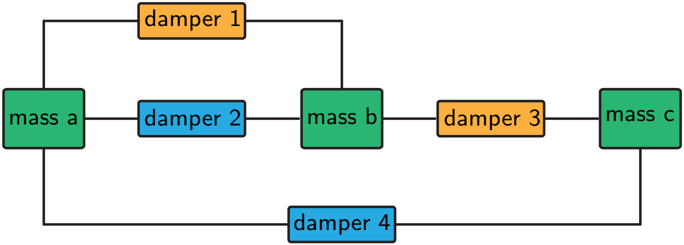
- Traffic networks
- Transportation and manufacturing networks
- Power networks
- Systems biology
- Electrical circuits
- Mechanical networks
- ...

There is need to reduce the large-scale complex model to a lower-dimensional one, but within the same class.

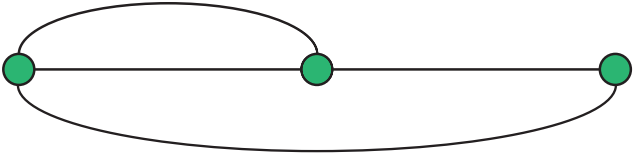
However, common model reduction techniques do not do this.

Outline

Mass-damper systems



mass-damper network



underlying graph

Associate masses to the nodes, and dampers to the edges.

Graph formulation

A graph consists of a set of n nodes, and a set of k edges (between the nodes). Endow the graph with an arbitrary orientation (sign convention): directed graph.

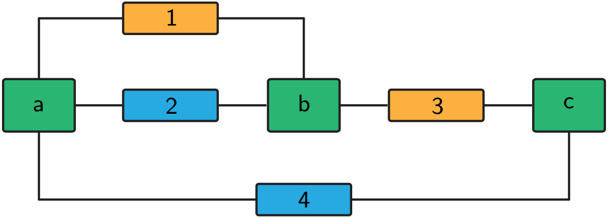
The directed graph is specified by its $n \times k$ incidence matrix D:
each column corresponds to an edge and contains one -1 (tail node) and one 1 (head node).

Basic property

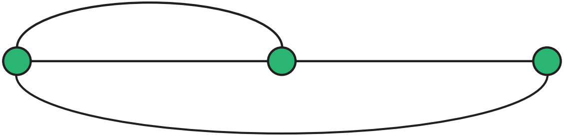
$$\mathbb{1}^T D = 0$$

where $\mathbb{1}$ is the vector of all ones. The graph is connected if and only if

$$\ker D^T = \operatorname{span} \mathbb{1}$$



mass-damper network



underlying graph

$$D = \begin{bmatrix} -1 & 1 & 0 & 1 \\ 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Dynamics of mass-damper systems

Elements of $D^T v \in \mathbb{R}^k$ are differences of velocities $v \in \mathbb{R}^n$ at the head and tail nodes of the edges. Thus

$$\dot{p} = -DRD^T v, \qquad p \in \mathbb{R}^n$$

- p vector of momenta of the n masses at the nodes
- $v := M^{-1} p \in \mathbb{R}^n$ vector of velocities of the masses
- M positive diagonal $n \times n$ mass matrix
- R positive diagonal $k \times k$ damping matrix
- $L := DRD^T$ a symmetric (weighted) Laplacian matrix

Can be immediately extended to mass-damper networks in $\mathbb{R}^3$ using Kronecker products with 3×3 identity matrix;
and further to damped multi-body systems.

Other examples of the same form

Hydraulic networks:

n fluid reservoirs with storage variables $s_i, i = 1, \cdots, n$.
Mass balance means

$$\dot{s} = Df,$$

where f_j is the flow through the j-th pipe, $j = 1, \cdots, k$.

Each storage variable s_i determines a pressure $\pi_i := \frac{s_i}{m_i}$.

Assume each flow f_j is proportional to the difference of the pressures at the head reservoir and the tail reservoir:

$$f_j = -r_j(\pi_h - \pi_t)$$

Leads again to the dynamics

$$\dot{s} = -DRD^T\pi, \qquad \pi = M^{-1}s$$

Other examples of the same form cont'd

RC-circuits: grounded capacitors with linear resistors

$$\dot{Q} = -DGD^T C^{-1} Q, \qquad Q \text{ charges},$$

G diagonal conductance matrix, and C diagonal capacitance matrix

Symmetric consensus dynamics:

Corresponds to mass-damper systems with unit masses:

$$\dot{x} = -DRD^T x = -Lx$$

Diagonal elements of R correspond to weights in consensus algorithm.

Nonlinear extensions

Mass-damper network with nonlinear dampers

$$\dot{p} = -D\mathcal{R}(D^T v), \qquad v = M^{-1}p$$

with $\mathcal{R} : \mathbb{R}^k \to \mathbb{R}^k$ diagonal damping mapping.

Further extension: hydraulic network with nonlinear damping of the pipes, as well as nonlinear pressure relations

$$\dot{s} = -D\mathcal{R}(D^T \pi), \qquad \pi = \frac{\partial H}{\partial s}(s)$$

with $H(s)$ the stored energy at the reservoirs with storage s.
Previous linear case $H(s) = \frac{1}{2}s^T M^{-1} s$.

Another generalization: chemical reaction networks

Detailed-balanced mass-action kinetics chemical reaction networks[1]

$$\dot{x} = -ZDRD^T \mathrm{Exp}\left(Z^T \frac{\partial G}{\partial x}(x)\right), \quad x \in \mathbb{R}^m_{>0} \text{ concentrations}$$

with Gibbs' free energy (w.r.t. thermodynamic equilibrium x^*)

$$G(x) = x^T \mathrm{Ln}\left(\frac{x}{x^*}\right) + (x^* - x)^T \mathbb{1}_m, \quad \frac{\partial G}{\partial x}(x) = \mathrm{Ln}\left(\frac{x}{x^*}\right)$$

The matrix $Z : \mathbb{R}^m \to \mathbb{R}^c$ describes the composition of the chemical complexes in the chemical species. $S = ZD$ is the stoichiometric matrix.

For single-species chemical reaction networks $Z = I$; thus reducing to

$$\dot{x} = -DRD^T \frac{x}{x^*}, \qquad x \in \mathbb{R}^m_{>0}$$

Identical to mass-damper system with $M := \mathrm{diag}\left(x_1^*, \cdots, x_n^*\right)$.

[1] AJvdS, Rao, Jayawardhana, SIAP2013.

Outline

Clustering

Consider mass-damper system

$$\dot{x} = -DRD^T M^{-1} x + Eu, \quad x \in \mathbb{R}^n, \; u \in \mathbb{R}^m$$

Main idea: clustering of nodes.

Consider a partition $\mathcal{P}$ of the node set into $\hat{n}$ disjoint cells $C_1, C_2, \cdots, C_{\hat{n}}$. This corresponds to an $n \times \hat{n}$ characteristic matrix P:
i-th column of P is the characteristic vector of cell i: the vector with 1 at the position of every node contained in cell C_i, and 0 elsewhere.

Example: Consider the mass-damper system, and cluster the masses a, b, c into $C_1 = \{a, b\}, C_2 = \{c\}$. Then

$$P = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$P^T D$ is composed of zero columns (corresponding to edges linking nodes within the same cell),
and a remaining incidence matrix $\hat{D}$ of the reduced graph with nodes given by the cells.

Example: Cluster the masses a, b, c into the two cells $C_1 = \{a, b\}, C_2 = \{c\}$. Then

$$P^T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} -1 & 1 & 0 & 1 \\ 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$P^T D = \begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 0_{2\times 2} & \hat{D} \end{bmatrix}$$

Reduced mass-damper system

The reduced system is defined as[2]

$$\dot{\hat{x}} = -(P^T D R D^T P)(P^T M P)^{-1}\hat{x} + P^T E u,$$

with $\hat{x} := P^T x \in \mathbb{R}^{\hat{n}}$ the clustered state vector.
Define the $\hat{n} \times \hat{n}$ diagonal matrix

$$\hat{M} := P^T M P,$$

$\hat{R}$ the truncation of R, and $\hat{E} := P^T E$.
The reduced system is again a mass-damper system

$$\dot{\hat{x}} = -\hat{D}\hat{R}\hat{D}^T \hat{M}^{-1}\hat{x} + \hat{E}u, \quad \hat{x} \in \mathbb{R}^{\hat{n}}$$

on the reduced graph with nodes being the cells, and edges between nodes in different cells. $\hat{x}_i$ is the total momentum of the masses in cell C_i.

[2] AJvdS, MTNS2014.

Petrov-Galerkin interpretation

Recall: A Petrov-Galerkin reduction of $\dot{x} = Ax$ is

$$\dot{\hat{x}} = W^T A V \hat{x},$$

where V and W are $n \times \hat{n}$ matrices such that $W^T V = I$.

The reduced mass-damper system is a Petrov-Galerkin reduction with

$$W := P, \quad V := MP\hat{M}^{-1} = MP(P^T MP)^{-1}$$

Key question: how to cluster?

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Node and edge dynamics

Consider the mass-damper system with node dynamics

$$\begin{aligned} \dot{x} &= -DRD^T M^{-1}x + DFu, \quad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m \\ &= -LM^{-1}x + DFu, \qquad L = DRD^T \ \text{Laplacian} \end{aligned}$$

Define $e = D^T M^{-1} x \ (= D^T v)$. Then

$$\begin{aligned} \dot{e} &= D^T M^{-1}\dot{x} = -D^T M^{-1} DRD^T M^{-1} x + D^T M^{-1} DFu \\ &= -L_e Re + L_e Fu, \qquad L_e := D^T M^{-1} D \ \text{edge Laplacian} \end{aligned}$$

Note that L_e is invertible iff the graph has no cycles; in which case the edge dynamics defines a gradient system, with inner product given by L_e^{-1}.

Balancing of edge dynamics

Consider the node dynamics with outputs

$$\begin{aligned} \dot{e} &= -L_e R e + L_e F u \\ y &= F^T e \end{aligned}$$

Typically columns of F are basis vectors corresponding to controlled edge flows, with conjugated outputs.

Main idea: Do balancing of the edge dynamics, and identify the edges which are least important. Then cluster the nodes that are only linked by unimportant edges.[3]

[3]Besselink, Sandberg, Johansson, Clustering-based model reduction of networked passive systems, IEEE-TAC, 2016.

The observability Gramian $\mathcal{O}$ and controllability Gramian $\mathcal{C}$ (in case there are no cycles!) are related by

$$\mathcal{C} = L_e \mathcal{O} L_e$$

and it follows that in balanced coordinates[4]

$$S^T L_e S = S^T D^T M^{-1} D S = I_k$$

Otherwise, in order to identify the least important edges, we have to rely on generalized observability and controllability Gramians;

see the paper by Besselink et al., also for error bounds. See also the work of Imura and co-workers.

[4]Scherpen & AJvdS, IFAC-WC 2011.

'Duality' between node and edge dynamics

Consider the node dynamics

$$\dot{x} = -LM^{-1}x, \quad L = DRD^T$$

and the edge dynamics

$$\dot{e} = -L_e Re, \quad L_e = D^T M^{-1} D$$

Consider a reduced node dynamics obtained by clustering as before

$$\dot{\hat{x}} = -\hat{L}\hat{M}^{-1}\hat{x}, \quad \hat{L} = \hat{D}\hat{R}\hat{D}^T$$

Then the corresponding edge dynamics

$$\dot{\hat{e}} = -\hat{L}_e \hat{R}\hat{e}, \quad \hat{L}_e = \hat{D}^T \hat{M}^{-1} \hat{D}$$

is such that $\hat{L}_e$ is the Schur complement of L_e with respect to the edges that are within cells!

Thus clustering of nodes corresponds to Kron reduction of edges !

A special class of partitions

Consider a partition P of the graph which is generalized equitable with respect to the weight matrices R and M in the following sense

$$L(\operatorname{im} P) \subset M(\operatorname{im} P)$$

Construct an $n \times (n - \hat{n})$ matrix S such that $\operatorname{im} S$ is orthogonal to $\operatorname{im} P$ with respect to the inner product defined by M, and the $n \times n$ stacked matrix $\begin{bmatrix} P & S \end{bmatrix}$ is invertible.
It follows that also $\operatorname{im} S$ is an invariant subspace of $L = DRD^T$, and

$$\begin{bmatrix} P^T \\ S^T \end{bmatrix} DRD^T \begin{bmatrix} P & S \end{bmatrix} = \begin{bmatrix} \hat{D}\hat{R}\hat{D}^T & 0 \\ 0 & L_S \end{bmatrix}$$

for a certain symmetric matrix L_S, with $\hat{D}, \hat{R}$ as before.

Generalized equitable partitions cont'd

It follows that the transformed state

$$z := \begin{bmatrix} P^T \\ S^T \end{bmatrix} x =: \begin{bmatrix} \hat{x} \\ x' \end{bmatrix}$$

satisfies the dynamics

$$\begin{bmatrix} \dot{\hat{x}} \\ \dot{x}' \end{bmatrix} = - \begin{bmatrix} \hat{D}\hat{R}\hat{D}^T & 0 \\ 0 & L_S \end{bmatrix} M_z^{-1} \begin{bmatrix} \hat{x} \\ x' \end{bmatrix} + \begin{bmatrix} P^T \\ S^T \end{bmatrix} Eu,$$

where M_z is the transformed diagonal matrix

$$M_z := \begin{bmatrix} P^T \\ S^T \end{bmatrix} M \begin{bmatrix} P & S \end{bmatrix},$$

whose left-upper block is equal to $\hat{M}$.
Hence in this case the reduced system is an invariant sub-dynamics of the full-order system.

H_2-error expression[5]

Take as output

$$y = R^{\frac{1}{2}} D^T M^{-1} x = R^{\frac{1}{2}} D^T v$$

Then the transfer matrices satisfy

$$\|G - \hat{G}\|_2^2 = \frac{1}{2} \sum_{i \in \mathcal{V}_c} \left(\frac{1}{m_i} - \frac{1}{M_{\hat{i}}} \right)$$

where $\mathcal{V}_c$ is the set of controlled nodes, and $M_{\hat{i}}$ is the total mass of the masses belonging to the same cell $C_{\hat{i}}$ as i.

[5] N. Monshizadeh, AJvdS, Structure-preserving model reduction of physical network systems by clustering, 53rd IEEE-CDC, 2014.

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Origin of Kron reduction

Consider an electrical resistor circuit, with internal and external nodes (terminals).
Relation between node voltage potentials V and node currents I is given by

$$\begin{bmatrix} I_e \\ 0 \end{bmatrix} = DRD^T \begin{bmatrix} V_e \\ V_i \end{bmatrix}$$

By eliminating V_i this leads to the Schur complement[6]

$$I_e = \hat{D}\hat{R}\hat{D}^T V_e$$

Externally equivalent resistor circuit without internal nodes.

[6] AJvdS, Systems & Control Lett., 2011.

Kron reduction for mass-damper systems

Split the vector p into p_s (slow) and p_f (fast)

$$p = \begin{bmatrix} p_s \\ p_f \end{bmatrix}, \quad p_s \in \mathbb{R}^{n_s}, p_f \in \mathbb{R}^{n_f}, n_s + n_f = n$$

Split the incidence matrix as

$$D = \begin{bmatrix} D_s \\ D_f \end{bmatrix}$$

leading to

$$\begin{aligned} \begin{bmatrix} \dot{p}_s \\ \dot{p}_f \end{bmatrix} &= -\begin{bmatrix} D_s R D_s^T & D_s R D_f^T \\ D_f R D_s^T & D_f R D_f^T \end{bmatrix} \begin{bmatrix} M_s^{-1} & 0 \\ 0 & M_f^{-1} \end{bmatrix} \begin{bmatrix} p_s \\ p_f \end{bmatrix} + \begin{bmatrix} E_s \\ E_f \end{bmatrix} u \\ y &= \begin{bmatrix} E_s^T & E_f^T \end{bmatrix} \begin{bmatrix} M_s^{-1} & 0 \\ 0 & M_f^{-1} \end{bmatrix} \begin{bmatrix} p_s \\ p_f \end{bmatrix} \end{aligned}$$

Reduction based on slow/fast assumption

Assuming that $\dot{p}_f = 0$ we may solve

$$0 = \dot{p}_f = -D_f R D_s^T M_s^{-1} p_s - D_f R D_f^T M_f^{-1} p_f + E_f u$$

for $M_f^{-1} p_f$, leading to reduced system in the slow variables

$$\begin{aligned} \dot{p}_s &= -[D_s R D_s^T - D_s R D_f^T (D_f R D_f^T)^{-1} D_s R D_f^T] M_s^{-1} p_s + \hat{E} u \\ y_s &= \hat{E}^T M_s^{-1} p_s \end{aligned}$$

The matrix

$$D_s R D_s^T - D_s R D_f^T (D_f R D_f^T)^{-1} D_s R D_f^T = \widehat{D} \widehat{R} \widehat{D}^T$$

is Schur complement of the Laplacian matrix $L = DRD^T$.
Disadvantage: even if L is sparse then its Schur complement is likely to be dense: reduction of number of nodes but increase of number of edges.

Outline

Non-symmetric flow Laplacians

Basic conservation (mass-balance) laws:

$$\dot{x} = Df + Eu, \qquad x \in \mathbb{R}^n,\ f \in \mathbb{R}^k,\ u \in \mathbb{R}^n$$

with f vector of flows through the edges, and u external node flows.

In many physical systems (assuming linearity)

$$f = -R\sigma, \quad \sigma = D^T \sigma_s, \quad \sigma_s = Px$$

For example, in mass-damper systems, with $x = p$

$$f = -RD^T v = -RD^T M^{-1} p$$

This leads to $\dot{x} = -DRD^T Px = -LPx$ as before.
However, e.g. in transportation networks, one may find the more general form $f = Kx$, with $k \times n$ nonnegative transportation matrix K, where K_{ji} is non-zero if there is a flow f_j along the j-th edge originating from node i given as

$$f_j = K_{ji} x_i$$

This leads to

$$\dot{x} = DKx + Eu = -Lx + Eu$$

where

$$L := -DK$$

is a flow-Laplacian matrix: nonnegative diagonal elements, nonpositive off-diagonal elements, and $\mathbb{1}^T L = 0$.

But L is not symmetric, and in general $L\mathbb{1} \neq 0$.

Key question: how to generalize the previous results to flow-Laplacian matrices L ?

Excursion to alternative formulations of flow networks

Consider

$$\dot{x} = Df + Eu,$$

and write

$$f = f^+ - f^-, \quad u = u^+ - u^-, \quad f^+, f^-, u^+, u^- \geq 0$$

Define the $n \times n$ matrix F with

$$F_{ij} = f^+_{(i,j)}, \quad F_{ji} = f^-_{(i,j)}$$

Then

$$\dot{x} = Df + Eu = D(f^+ - f^-) + E(u^+ - u^-) = F^T \mathbb{1} - F \mathbb{1} + Eu^+ - Eu^-$$

Demand constraints on total outflow $z := Df^- + u^- \leq \phi(x)$:

$$\dot{x} = u^+ - z + R^T z = u^+ - (I - R^T)\Gamma\phi(x), \quad \gamma \in [0,1]$$

Transformation of L by Kirchhoff's Matrix Tree theorem

Consider a connected component of the graph, i.e., $\ker L$ is 1-dimensional.

Figure: Gustav Robert Kirchhoff, 1824 - 1887,
Über die Auflösung der Gleichungen, auf welche man bei der Untersuchung der Linearen Verteilung galvanischer Ströme geführt wird,
Ann. Phys. Chem. 72, 497–508, 1847.

Then a spanning vector $\rho \in \mathbb{R}^n_{\geq 0}$ for $\ker L$ can be computed by

Kirchhoff's Matrix Tree theorem

(in current form due to e.g. Tutte, 1917 - 2002)

Kirchhoff's Matrix Tree theorem continued

The adjoint matrix $\mathrm{adj}(L)$ consisting of co-factors C_{ij} of L satisfies

$$L \cdot \mathrm{adj}(L) = (\det L) I_n = 0$$

Furthermore, since $\mathbb{1}^T L = 0$ the sum of the rows of L is zero, and hence the co-factors C_{ij} do not depend on i; implying that

$$C_{ij} = \rho_j, \quad j = 1, \cdots, n$$

Therefore by defining $\rho := (\rho_1, \cdots, \rho_n)^T$

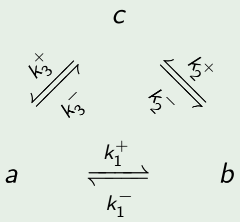
$$L\rho = 0$$

Furthermore, ρ_i is equal to

the sum of the products of weights of all the spanning trees directed towards vertex i.

In particular, $\rho_j \geq 0$, $j = 1, \cdots, n$.
$\rho \neq 0$ if and only if there is a spanning tree, and $\rho \in \mathbb{R}^n_+$ if and only if the graph is strongly connected.

Example (Cyclic transportation network)



$$L = \begin{bmatrix} k_1^+ + k_3^- & -k_1^- & -k_3^+ \\ -k_1^+ & k_1^- + k_2^+ & -k_2^- \\ -k_3^- & -k_2^+ & k_3^+ + k_2^- \end{bmatrix}$$

By Kirchhoff's Matrix Tree theorem

$$\rho = \begin{bmatrix} k_2^+ k_3^+ + k_1^- k_3^+ + k_1^- k_2^- \\ k_1^+ k_3^+ + k_1^+ k_2^- + k_2^- k_3^- \\ k_1^+ k_2^+ + k_2^+ k_3^- + k_1^- k_3^- \end{bmatrix}$$

In case all the connected components of the graph are strongly connected we can thus define the scaling transformation

$$\Pi := \operatorname{diag}(\rho_1, \cdots, \rho_n)$$

and rewrite

$$\dot{x} = -Lx = -L\Pi\Pi^{-1}x = -\mathcal{L}\Pi^{-1}x$$

where now $\mathcal{L} := L\Pi$ is balanced, that is

$$\mathbb{1}^T\mathcal{L} = 0, \quad \mathcal{L}\mathbb{1} = 0$$

This also implies

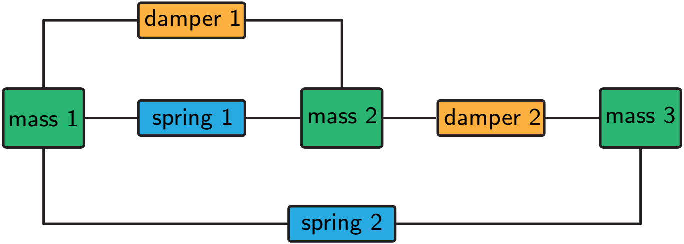
$$\mathcal{L} + \mathcal{L}^T \geq 0$$

For balanced Laplacians the same clustering procedure as in the symmetric case can be employed.
In particular, for any partition matrix P the reduced matrix $P^T\mathcal{L}P$ is still a balanced Laplacian matrix.

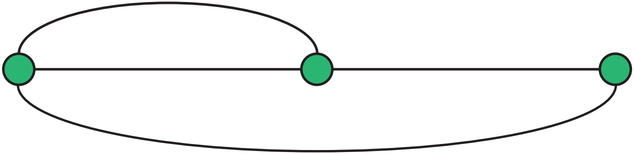
Outline

Mass-spring-damper systems

Nodes again correspond to masses, but edges to dampers and springs.



(a)



(b)

Partition $D = \begin{bmatrix} D_d & D_s \end{bmatrix}$:

D_d damper incidence matrix, D_s spring incidence matrix

Dynamics of the mass-spring-damper system takes the form

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & D_s^T \\ -D_s & -D_d R D_d^T \end{bmatrix} \begin{bmatrix} Kq \\ M^{-1}p \end{bmatrix}$$

$\frac{1}{2}p^T M^{-1} p + \frac{1}{2} q^T K q$ kinetic plus potential energy

Remark

Contrary to multi-agent systems the state is not only associated to the nodes (p), but also to the edges (q).
Typical phenomenon in many physical network systems.

Reduced mass-spring-damper system

Similarly as before define the clustered momenta vector

$$\hat{p} := P^T p$$

Reduced system on the reduced graph is again mass-damper system

$$\begin{bmatrix} \dot{\hat{q}} \\ \dot{\hat{p}} \end{bmatrix} = \begin{bmatrix} 0 & \hat{D}_s^T \\ -\hat{D}_s & -\hat{D}_d \hat{R} \hat{D}_d^T \end{bmatrix} \begin{bmatrix} \hat{K}\hat{q} \\ \hat{M}^{-1} p \end{bmatrix},$$

where $\hat{q}$ corresponds to the edges which survive in the reduced model.

Error bounds for clustering of the nodes have not yet been studied. Can we also cluster the edges?

Error expressions can be again derived whenever P is generalized equitable, but now both for the damper and the spring graph; see N. Monshizadeh, AJvdS, 53rd IEEE-CDC, 2014.

Swing equation model of power networks

$$
\begin{aligned}
\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} &= \begin{bmatrix} 0 & D^T \\ -D & -A \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q}(q,p) \\ \frac{\partial H}{\partial p}(q,p) \end{bmatrix} + \begin{bmatrix} 0 \\ u \end{bmatrix}, \quad p = M\omega \\
y &= \frac{\partial H}{\partial p}(q,p) = \omega
\end{aligned}
$$

with q phase differences across the transmission lines, ω frequency deviations at the nodes, and u generated/consumed power, $A > 0$ diagonal damping matrix, and Hamiltonian ('total shifted energy')

$$
H(q,p) = \frac{1}{2} p^T M^{-1} p - \mathbb{1}^T \Gamma \operatorname{Cos} q
$$

Outline

Conclusions and Outlook

- Model reduction of mass-damper-like systems by clustering.
- Base clustering on balancing of edge dynamics.
- Generalized equitable partitions.
- Structure-preserving model reduction based on Kron reduction.
- For mass-spring-damper case there are also state variables associated to the edges; contrary to the standard paradigm of multi-agent systems.
- Transformation of flow-Laplacians to balanced Laplacians for strongly connected components.
- Applications to metabolic reaction networks and power networks.

Some key references

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